Calculation of spectra of turbulence in the energy-containing and inertial ranges

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(Received 14 August 1997; revised manuscript received 2 April 1998)

A statistical theory for the power law stage of freely decaying homogeneous and isotropic developed turbulence is proposed. Attention is focused on the velocity field statistics in the energy-containing and inertial scales. The kinetic energy spectrum $E(k,t)$ and energy transfer spectrum $T(k,t)$ are calculated as functions of wave number k and decay time t. The scaling properties of the spectra of the stationary model of the randomly stirred fluid have been chosen as the starting point for the approximate derivation of time-dependent spectra $E(k,t)$ and $T(k,t)$. The stationary model analyzed by means of the renormalization group and short-distance expansion methods has provided the spectra $E(k)=C_K\varepsilon^{2/3}k^{-5/3}\mathcal{F}(kl)$ [where C_K is the Kolmogorov constant and $\mathcal{F}(kl)$ is a function] and $T(k)\propto\varepsilon k^{-1}\psi^{\{\mathcal{F}\}}(kl)$ [where $\psi^{\{\mathcal{F}\}}(kl)$ is functionally dependent on $\mathcal{F}$]. The characteristic length scale of these spectra defined from the mean square root velocity u and mean energy dissipation ε is the von Kármán scale $l=u^3/\varepsilon$. We have assumed that l, u, and ε as well as $E(k)$ and $T(k)$ are no longer constants but unknown functions of t. Scaling forms constructed in this way are consistent with the basic assumption of George's closure [W. M. George, Phys. Fluids A **4**, 1492 (1992)]. Power decay laws for $\varepsilon(t)$, $l(t)$, $u(t)$ and the constituent integro-differential equation for the scaling function $F(kl(t))=E(k,t)/C_K\varepsilon^{2/3}k^{-5/3}$ have been obtained using the equation of the spectral energy budget. The equation for $F(kl(t))$ has been investigated numerically for the three-dimensional system with Saffman's invariant [P. G. Saffman, J. Fluid Mech. **27**, 581 (1967); Phys. Fluids **10**, 1349 (1967)]. The calculated longitudinal energy spectrum has been compared with the available experimental data. [S1063-651X(98)00210-4]

PACS number(s): 47.27.Gs, 47.27.Jv, 11.10.Gh

I. INTRODUCTION

The Kolmogorov's universality hypothesis [1] supposing that the inertial range statistics of strongly developed turbulence is independent of the geometry of boundary conditions and the dynamics of dissipation scales has received remarkable experimental support [2,3]. To establish the validity of the phenomenologically obtained $-\frac{5}{3}$ law of the kinetic energy spectrum the renormalization group (RG) method has been applied [4] in the framework of Wyld's statistical model of the randomly stirred fluid [5]. Later development of the different RG variants applied to the stochastic hydrodynamics [6–8] was stimulated by the effort to find a well-founded calculation of the Kolmogorov constant.

It is a widespread opinion [2] that spectral properties of the energy-containing scales are nonuniversal, i.e., the lowest bound of a spectrum evolves under the action of anisotropy and finite-size effects. Even the classic works have indicated that it is only a very crude assumption ignoring the scaling features of the statistics of the energy-containing range. A more profound understanding of the universality was developed by George [9]. His analysis is based on the properties of the spectra measured at the intermediate distances behind the stirring grid (i.e., at the intermediate decay stages), where integral quantities such as the kinetic energy or dissipation rate satisfy power laws in time. By indicating the presence of a variety of possible scaling forms depending on the initial form of the spectrum, George has postulated a more general picture of the universality. The theory [10] based on the eddy damped quasinormal Markovian (EDQNM) approximation also supports the idea of the self-similarity within the energy-containing range. The remarkable difference between the above and the calculation presented here is that nonuniversal initial and late time stages of decay are fully inhibited in our theory. This property is a direct consequence of the fact that self-similarity of the spectra [see Eq. (4.2) below] is expected from the initial stages of the model formulation.

At the starting point of the study we shall employ the results accumulated previously for the randomly forced Navier-Stokes equation. In the stochastic formulation the additive Gaussian forcing is applied to model the unpredictable evolution of strong turbulent flows. In comparison with the variants of the stationary stochastic models exclusively focused on the statistics of the inertial scales, a modified definition of the pair correlation function of the random force is presented in our formulation. The main motivation for this generalization is our goal to study the scaling behavior of the spectra within the energy-containing range.

The present paper is organized as follows. In Sec. II we review the equations of the past grid turbulence. In Sec. III we summarize the quantum-field RG results obtained previously for the energy spectrum and energy transfer of the inertial range. The extension of their forms towards the energy-containing range is considered.

These forms represent the starting point of the decay analysis. In Sec. IV the model of decay is introduced and an integro-differential equation for the scaling function is derived. The numerical method for calculation of the scaling function with special emphasis on the asymptote of the inertial scales is discussed in Sec. V. Two different parametrizations of the scaling function in Sec. VI are suggested for the three dimensional decay with Saffman's invariant [see Eq. (4.36) below]. The limitations of the model are discussed in Sec. VII. In Appendix A the form of the energy transfer function is derived. In Appendix B details necessary to overcome the problems of the numerical integration are presented.

II. BASIC EQUATIONS OF THE PAST GRID EVOLUTION

The equation for the energy spectral budget plays a central role in the description of the strongly developed turbulence. If isotropy of the statistics, incompressibility of the fluid, and the absence of the external energy sources are supposed, the energy current conservation in the wave-number space is expressed by the equation

$$\partial_t E(k,t) = T(k,t) - 2\nu_0 k^2 E(k,t), \tag{2.1}$$

where

$$E(k,t) = \frac{1}{2} k^{d-1} S_d \int \frac{d^d \mathbf{x}_1}{(2\pi)^d} e^{-i\mathbf{k}\cdot\mathbf{x}_{12}} \langle \mathbf{v}(\mathbf{x}_1,t)\cdot\mathbf{v}(\mathbf{x}_2,t)\rangle,$$

$$T(k,t) = -k^{d-1} S_d \int \frac{d^d \mathbf{x}_1}{(2\pi)^d} e^{-i\mathbf{k}\cdot\mathbf{x}_{12}} \times \langle \mathbf{v}(\mathbf{x}_1,t)[\mathbf{v}(\mathbf{x}_2,t)\cdot\boldsymbol{\nabla}_{\mathbf{x}_2}]\mathbf{v}(\mathbf{x}_2,t)\rangle \tag{2.2}$$

are the energy spectrum and energy transfer, respectively; $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the area of the d-dimensional sphere of unit radius, $\Gamma(x)$ is the Gamma function, $\mathbf{x}_{12} \equiv \mathbf{x}_1 - \mathbf{x}_2$, and ν_0 is the kinematic viscosity. The angular brackets denote the statistical average over the realizations of the velocity fluctuations at the fixed time t. One can easily verify that the energy transfer fulfills the integral identity

$$\int_0^\infty dk\ T(k,t) = 0, \tag{2.3}$$

which leads to the equation for total energy conservation

$$\partial_t \mathcal{E} = -\varepsilon. \tag{2.4}$$

Here the mean kinetic energy $\mathcal{E}(t)$ and mean energy dissipation rate $\varepsilon(t)$ are defined by the integrals

$$\mathcal{E}(t) = \int_0^\infty dk\ E(k,t), \quad \varepsilon(t) = 2\nu_0 \int_0^\infty dk\ k^2 E(k,t). \tag{2.5}$$

It is well known that Eqs. (2.1), (2.4), and (2.5) may be transferred to the experimentally most easily accessible case of the past grid turbulence if Taylor's concept of frozen turbulence [2,11] is well justified. In accord with this concept, in the statistically homogeneous regions of the past grid flow, the time of decay t can be simply related to the distance measured in a streamwise direction from the position of the stirring grid.

III. STOCHASTIC MODEL OF STATIONARY ISOTROPIC TURBULENCE

Statistical information straightforwardly obtainable from Eq. (2.1) is incomplete due to the standard closure problem. The classical attempt to solve this problem consists of the search for an appropriate phenomenological relation between T and E [2] or the search for the relation between statistical moments of interest. The relations evaluated indirectly within the stochastic models [5–8] are considered less phenomenological due to significant elimination of empirical inputs, which are used to adjust the model parameters.

The basic equation that mimics the statistics of strong turbulence is the randomly forced Navier-Stokes equation

$$\partial_\tau v_j = \nu_0 \nabla^2 v_j - \sum_s^d \mathcal{P}_{js}(\boldsymbol{\nabla})(\mathbf{v}\cdot\boldsymbol{\nabla})v_s + f_j, \quad \boldsymbol{\nabla}\cdot\mathbf{v} = \boldsymbol{\nabla}\cdot\mathbf{f} = 0, \tag{3.1}$$

where $\mathcal{P}_{js}(\mathbf{k}) = \delta_{js} - k_j k_s/k^2$ is a transverse projection operator ensuring the incompressibility of the fluid. Two different time variables (t and τ) were introduced to distinguish between formulas corresponding to the stationary and decay case, respectively.

Following the tradition of the stochastic models of the turbulence, we assume that the statistics of the external random force $\mathbf{f}(\mathbf{x},\tau)$ is isotropic and Gaussian. It is completely determined by the averages

$$\langle f_j(\mathbf{x},\tau)\rangle = 0,$$

$$\langle f_j(\mathbf{x}_1,\tau_1) f_s(\mathbf{x}_2,\tau_2)\rangle = \delta(\tau_{12})\mathcal{P}_{js}(\boldsymbol{\nabla}_{\mathbf{x}_{12}})\mathcal{C}(|\mathbf{x}_{12}|), \tag{3.2}$$

where $\tau_{12} = \tau_1 - \tau_2$ and

$$\mathcal{C}(|\mathbf{x}|) = \int \frac{d^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} D^{\{\mathcal{F}\}}(k) = \frac{x^{(2-d)/2}}{(2\pi)^{d/2}} \int_0^\infty dk\ k^{d/2} J_{(d-2)/2}(kx) D^{\{\mathcal{F}\}}(k). \tag{3.3}$$

Here J is the Bessel function. The forcing spectrum $D^{\{\mathcal{F}\}}(k)$ is defined by

$$D^{\{\mathcal{F}\}}(k) = \bar{D}\mathcal{F}(kl)k^{-d}, \tag{3.4}$$

where $\bar{D}$ is the amplitude of the forcing correlations proportional to the mean injection rate of energy, which is equal to the mean energy dissipation rate ε.

The combination $\bar{D}/\nu_0^3$ plays the role of the coupling constant in the perturbative approach [4,12] and l is the typical length of the energy-containing range. In Sec. IV we will show that this scale can be associated with the von Kármán length scale. We assume that the region $k \gg 1/l$ corresponds to the inertial range. The explicit form of the function $\mathcal{F}(kl)$ is not specified yet and the forcing extension caused by the introduction of $\mathcal{F}(kl)$ requires some additional remarks. The single point correlation function $\int_{-\infty}^{\infty} d\tau_1 \langle \mathbf{f}(\mathbf{x},\tau_1)\cdot\mathbf{f}(\mathbf{x},\tau_2)\rangle$ is proportional to the energy injection rate $\mathcal{C}(0) \propto \int_0^\infty dk\ k^{d-1} D^{\{\mathcal{F}\}}(k)$. The argument of the integral $\mathcal{C}(0)$ represents the contributions of the separate velocity modes in the full energy supply of the forcing. For the scales that are comparable to l the function $\mathcal{F}(kl)$ reflects the details of the forcing mechanism. In spite of this ambiguity, it is reasonable to expect that a maximum of $k^{d-1}D^{\{\mathcal{F}\}}(k)=\bar{D}\mathcal{F}(kl)/k$ should be located at $k\propto 1/l$. The definition of the forcing correlator (3.4) can be regarded as a relation generalizing the basic k^{-d} form of the forcing spectrum [6,13] or its extension $k^4[k^2+(1/l)^2]^{-2}k^{-d}$ [see also Eq. (6.3) in Sec. VI] introduced in [4,8]. The term $1/l$ from the last expression plays the role of the infrared mass (when using the terms of the quantum field models). The corresponding energy contribution $k^{d-1}D^{\{\mathcal{F}\}}(k)\propto k^{-1}(kl)^4/[1+(kl)^2]^2$ has a maximum at the wave number $\sqrt{3}/l$. In Refs. [6,13], the use of forcing with the correlator $\sim k^{-d}$ [which is the massless limit of Eq. (3.4)] was motivated by the intention to apply the RG method to describe the inertial range statistics. Therefore, the additional limitation to the form of $\mathcal{F}(kl)$ is the assumption that within the inertial range ($kl\gg 1$) the spectrum $D^{\{\mathcal{F}\}}(k)$ approaches the tail k^{-d}. This assumption is compatible with the normalization

$$\lim_{x\to\infty}\mathcal{F}(x)=1. \tag{3.5}$$

The system of the velocity correlation functions is completely determined by the stochastic model represented by Eqs. (3.1), (3.3), and (3.4). Naturally, the information implemented by $\mathcal{F}(kl)$ will be reflected in the properties of the correlation functions. Only in the special case $\mathcal{F}=1$ is the pure inertial range statistics generated.

To solve the complicated problem of the calculation of the spectral characteristics $E(k)$ and $T(k)$, advanced theoretical tools utilizing the quantum field RG [4,14] and so-called *short-distance expansion* techniques [12,15] have been applied. The application was inspired by the general approaches developed in the quantum field theory and the theory of critical phenomena [16].

In [4,8,14] it was shown that the forcing $\bar{D}k^{-d}$ induces the energy spectrum of the form

$$E(k)=c_E\bar{D}^{2/3}k^{-5/3}, \tag{3.6}$$

where

$$c_E=\frac{(d-1)(g_*)^{1/3}S_d}{4(2\pi)^d},\quad g_*=\frac{16(d+2)(2\pi)^d}{3(d-1)S_d}. \tag{3.7}$$

The dimensionless parameter g_* occurring here was fixed by the RG transformation. The presence of the extended forcing (3.4) is reflected in the energy spectrum

$$E(k)=c_E\bar{D}^{2/3}\mathcal{F}(kl)k^{-5/3} \tag{3.8}$$

of the inertial and energy-containing scales. The only difference between Eqs. (3.6) and (3.8) is the presence of the function $\mathcal{F}$. Unfortunately, already in the simplest case of the pure inertial range [where $\mathcal{F}(\chi)=1\,\forall\chi$], problems appear in the analysis of the triple velocity correlation function [15]. This function is related to the energy transfer via an integral formula. In Appendix A the reader can find the salient details of its derivation for the extended forcing. The resulting form of $T(k)$ is given by

$$T(k)=c_T\bar{D}k^{-1}\psi^{\{\mathcal{F}\}}(kl),\quad c_T=\frac{g_*S_d}{2(2\pi)^d}, \tag{3.9}$$

where

$$\psi^{\{\mathcal{F}\}}(\chi)=\tilde{c}_T\int_\Delta dq\,dp\,R(q,p;\chi),\quad \tilde{c}_T=\frac{S_{d-1}}{2^{d+1}(2\pi)^d}, \tag{3.10}$$

is the dimensionless scaling function of dimensionless variable $\chi=kl$ and

$$\begin{aligned}R(q,p;\chi)=K(p,q)\{&\mathcal{F}(p\chi)\mathcal{F}(q\chi)(pq)^{-d-2/3}\\&\times[Q(p,q)+Q(q,p)]-\mathcal{F}(\chi)\\&\times[\mathcal{F}(p\chi)p^{-d-2/3}Q(p,q)\\&+\mathcal{F}(q\chi)q^{-d-2/3}Q(q,p)]\},\end{aligned} \tag{3.11}$$

$$K(p,q)=\frac{[2(q^2+q^2p^2+p^2)-(1+q^4+p^4)]^{(d-1)/2}}{pq(1+q^{2/3}+p^{2/3})}, \tag{3.12}$$

$$Q(p,q)=p^4-p^2+(d-1-p^2)q^2. \tag{3.13}$$

In the integral (3.10) Δ denotes the domain of integration

$$\Delta\equiv\{(q,p);q\geq 0,|1-q|\leq p\leq 1+q\}. \tag{3.14}$$

The inertial range critical exponents $-\frac{5}{3}$ for $E(k)$ [and the exponent -1 for the k dependence of $T(k)$] were perturbatively calculated exactly [12,14] using the ϵ-expansion technique (see, e.g., [16]). The stationary form of $T(k)$ as given by Eqs. (3.9)–(3.13) is consistent with the EDQNM model [7,17,18]. The principal difference between the form of energy transfer resulting from the Markovian nature and the similar form given by Eqs. (3.9)–(3.13) is the presence of the function $\mathcal{F}$ and the presence of the constant parameter c_T, which is fixed by the RG flow. Since the final treatment of the model needs the support of the numerical analysis, in Appendix B some of the details are discussed that allow for the efficient calculation of two dimensional integrals of the type (3.10).

IV. MODEL OF THE TURBULENCE DECAY: POWER FORM OF THE TIME EVOLUTION

The decay of the freely evolving turbulence can be modeled by assuming that the role of the energy input is played by the term $\partial_t E$. Since we are looking for $E(k,t)$ and $T(k,t)$ in the restricted region of the inertial and energy-containing scales, the viscous effects can be neglected and the equation for the spectral budget is applicable in the truncated form $\partial_t E=T$.

Assuming low speed of the time variations and partial persistence of the distribution of the spatial turbulent structures, the decay spectra can be constructed using modified stationary forms [see Eqs. (3.8) and (3.9)]. Applying the substitutions

$$\mathcal{F}\to F, \quad l\to l(t), \quad \overline{D}\to\overline{D}(t), \tag{4.1}$$

we have obtained

$$E(k,t)=c_E[\overline{D}(t)]^{2/3}F(kl(t))k^{-5/3},$$

$$T(k,t)=c_T\overline{D}(t)k^{-1}\psi^{\{F\}}(kl(t)). \tag{4.2}$$

To distinguish between the stationary and decay situations, two different symbols $\mathcal{F}$ and F [as well as the functionals $\psi^{\{\mathcal{F}\}}(kl(t))$ and $\psi^{\{F\}}(kl(t))$] have been introduced; the symbol $F(kl)$ plays the role of the scaling function. The underlying problem associated with our approximate construction is estimating the bounds of the admissible domain in the wave-number time or parametric spaces (see Sec. VII). Regardless of the detailed structure of $\psi^{\{F\}}(\chi)$, Eq. (4.2) is compatible with the self-similar relations postulated by von Kármán and Howarth [19] and George [9].

In this section we present the derivation of the time dependences $\overline{D}(t),l(t)$ as well as the derivation of a system of equations for the scaling function $F(\chi)$. If the time-dependent scaling forms (4.2) are inserted into the inviscid variant of Eq. (2.1), the following equation can be obtained:

$$\phi_1(t)\chi^{-2/3}\left[1+\phi_2(t)\chi\frac{d}{d\chi}\right]F(\chi)=\psi^{\{F\}}(\chi), \tag{4.3}$$

where

$$\phi_1(t)=\frac{2c_E}{3c_T}[\overline{D}(t)]^{-1/3}[l(t)]^{2/3}\partial_t\ln\overline{D}(t), \tag{4.4}$$

$$\phi_2(t)=\frac{3}{2}\,\frac{\partial_t\ln l(t)}{\partial_t\ln\overline{D}(t)}. \tag{4.5}$$

The differentiation of Eq. (4.3) with respect to t gives rise to the auxiliary equation that allows us to determine the conditions of its solubility. The comprehensive analysis taking into account the auxiliary equation and Eq. (4.3) shows that the most informative physical solution is obtained when

$$\phi_1(t)=c_1, \quad \phi_2(t)=c_2 \tag{4.6}$$

(c_1 and c_2 are some integration constants) and

$$\hat{L}F(\chi)=\psi^{\{F\}}(\chi), \quad \hat{L}\equiv c_1\chi^{-2/3}\left[1+c_2\chi\frac{d}{d\chi}\right]. \tag{4.7}$$

From the normalization (3.5) and Eq. (4.7) we conclude that $\hat{L}F|_{F=1}$ asymptotically vanishes,

$$\hat{L}F|_{F=1}=c_1\chi^{-2/3} \quad \text{for } c_1\neq 0, \quad \chi\gg 1, \tag{4.8}$$

whereas [3]

$$\psi^{\{F\}}(\chi)=0 \quad \text{for } F=1, \tag{4.9}$$

i.e., when the energy spectrum acquires the pure Kolmogorov form. This finding is consistent with the weakened variant of Eq. (4.7),

$$\lim_{\chi\to\infty}[\hat{L}F(\chi)-\psi^{\{F\}}(\chi)]=0. \tag{4.10}$$

A more restrictive condition

$$\lim_{\chi\to\infty}\chi^{2/3}[\hat{L}F(\chi)-\psi^{\{F\}}(\chi)]=0 \tag{4.11}$$

is used in Sec. V to control the coincidence of the leading asymptotic behavior of $\psi^{\{F\}}(\chi)$ and next to leading order asymptotics of $F(\chi)$. In accord with the assumption [19] Eqs. (4.4)–(4.6) have the solution

$$\overline{D}(t)=D_0t^{\alpha_D}, \quad l(t)=l_0t^{\alpha_l}. \tag{4.12}$$

The amplitudes D_0,l_0 and exponents α_l,α_D are related to the separation constants c_1,c_2 via the relations

$$D_0=l_0^2\left(\frac{2c_E\alpha_D}{3c_Tc_1}\right)^3, \quad \alpha_l=\frac{6c_2}{4c_2-3}, \quad \alpha_D=\frac{9}{4c_2-3}. \tag{4.13}$$

When Eqs. (3.8), (4.12), and (4.13) are inserted into Eqs. (2.4) and (2.5) one can obtain the decay laws of the kinetic energy and energy dissipation

$$\mathcal{E}(t)=\mathcal{E}_0t^{\alpha_\mathcal{E}}, \quad \alpha_\mathcal{E}=\frac{2(\alpha_D+\alpha_l)}{3}, \tag{4.14}$$

$$\mathcal{E}_0=\frac{4c_E^3\alpha_D^2l_0^2}{9c_T^2c_1^2}I^{\{F\}}, \tag{4.15}$$

$$\varepsilon(t)=-\alpha_\mathcal{E}\mathcal{E}_0t^{\alpha_\mathcal{E}-1}, \tag{4.16}$$

$$I^{\{F\}}\equiv\int_0^\infty dx\ x^{-5/3}F(x). \tag{4.17}$$

Similarly, for the mean square root velocity $u(t)$ one obtains

$$u(t)=\sqrt{\frac{2}{d}\,\mathcal{E}(t)}=u_0t^{\alpha_\mathcal{E}/2}, \quad u_0=\sqrt{\frac{2}{d}\,\mathcal{E}_0}. \tag{4.18}$$

Now we demonstrate how the separation constant c_2 introduced in Eq. (4.6) reflects the structure of $F(\chi)$ for $\chi\to 0$. At the end of the section, the connection between c_1 and Kolmogorov constant C_K is provided.

Following [3], the assumption about the large-scale structure of the turbulence has been supplemented

$$\lim_{k\to 0}\frac{E(k,t)}{k^\alpha}=\Lambda_\alpha=\text{const}>0, \quad \alpha>0, \tag{4.19}$$

where Λ_α and α are new parameters. The choice of α implies the selection of the flow invariant Λ_α [see Eqs. (4.31) and (4.32) below]. From Eqs. (3.8) and (4.19) it follows that

$$F(\chi)=c_F\chi^{(3\alpha+5)/3} \quad \text{as } \chi\ll 1, \tag{4.20}$$

$$\Lambda_\alpha = c_F c_E \overline{D}^{2/3} l^{(3\alpha+5)/3}, \tag{4.21}$$

where c_F is a constant. The requirement of the time invariance of Λ_α and Eqs. (4.12) and (4.21) lead to the connection of the exponents $\alpha_D = -(\alpha_l/2)(3\alpha+5)$. Then, using Eqs. (4.13), with the help of Eq. (4.7) we obtain

$$c_2 = -\frac{3}{3\alpha+5}, \quad \hat{L}F|_{F=c_F x^{(3\alpha+5)/3}} = 0. \tag{4.22}$$

From the relations (4.13), (4.14), and (4.22) it follows that

$$\alpha_l = \frac{2}{\alpha+3}, \quad \alpha_D = -\frac{3\alpha+5}{\alpha+3}, \quad \alpha_{\mathcal{E}} = 1+\alpha_D = -\frac{2(\alpha+1)}{\alpha+3}. \tag{4.23}$$

In analogy with the stationary case, the Kolmogorov constant has been defined by the relation

$$C_K = \frac{E(k,t)}{[\varepsilon(t)]^{2/3} k^{-5/3} F(kl)}. \tag{4.24}$$

The normalization (3.5) implies that the inertial range spectrum $C_K \varepsilon^{2/3} k^{-5/3}$ is achieved for $k \gg 1/l$. Equations (4.2) and (4.24) can be understood as two different representations of the same $E(k,t)$ spectrum. Their comparison and application of Eqs. (3.9), (4.4), (4.6), (4.13), and (4.14) yield

$$C_K = c_E \left(\frac{\overline{D}}{\varepsilon} \right)^{2/3} = c_E \left(-\frac{D_0}{\alpha_{\mathcal{E}} \mathcal{E}_0} \right)^{2/3} = c_E \left(-\frac{2\alpha_D}{3\alpha_{\mathcal{E}} c_T c_1 I^{\{F\}}} \right)^{2/3}. \tag{4.25}$$

The time independence of C_K stems from Eq. (4.23). From Eqs. (2.5) and (4.24) we have

$$\mathcal{E} = C_K I^{\{F\}} \varepsilon^{2/3} l^{2/3}. \tag{4.26}$$

By the use of Eqs. (4.18) and (4.26) we obtain

$$l = \frac{u^3}{\varepsilon} \left(\frac{2 C_K I^{\{F\}}}{d} \right)^{3/2}. \tag{4.27}$$

The last formula resembles the established definition of the von Kármán length scale

$$l_{\mathcal{K}} = \frac{u^3}{\varepsilon}. \tag{4.28}$$

Therefore, it is convenient to suggest the calibration $l = l_{\mathcal{K}}$ to give rise to the connection

$$I^{\{F\}} = \frac{d}{2C_K}. \tag{4.29}$$

Substitution of this form into Eq. (4.25) relates c_1 and C_K:

$$c_1 = \frac{\tilde{c}_1}{\sqrt{C_K}}, \quad \text{where } \tilde{c}_1 = \frac{4(1-\alpha_{\mathcal{E}})(c_E)^{3/2}}{3d\alpha_{\mathcal{E}} c_T}. \tag{4.30}$$

From Eqs. (4.25), (4.28), and (4.23) we rewrite Eq. (4.21) in terms of ε, u, and l:

$$\Lambda_\alpha = c_F C_K \varepsilon^{2/3} l^{(3\alpha+5)/3} = c_F C_K u^2 l^{1+\alpha} = c_F C_K u_0^2 l_0^{1+\alpha}. \tag{4.31}$$

The characteristics that are essential for the construction of the decay phenomenology are the dimensional integrals [2]

$$\mathcal{I}_\alpha = \int_0^\infty dx\, x^\alpha B_{LL}(x,t). \tag{4.32}$$

Here $B_{LL}(x,t)$ is the longitudinal velocity pair correlation function defined by

$$B_{LL}(|\mathbf{x}_{12}|,t) = \sum_{j,s}^{d} \langle v_j(\mathbf{x}_1,t) v_s(\mathbf{x}_2,t) \rangle \frac{(\mathbf{x}_{12})_j (\mathbf{x}_{12})_s}{|\mathbf{x}_{12}|^2}. \tag{4.33}$$

It is connected to the $E(k,t)$ spectrum by means of the relation

$$B_{LL}(x,t) = \int_0^\infty dk\, E(k,t) B_L^E(kx), \tag{4.34}$$

where $B_L^E(y) = 2(\sin y - y\cos y)/y^3$ in the case $d=3$. The substitution of Eq. (4.24) into Eq. (4.34) gives

$$B_{LL}(x,t) = [u(t)]^2 \tilde{B}_L\left(\frac{x}{l(t)} \right),$$

$$\tilde{B}_L(y) = C_K \int_0^\infty dx\, B_L^E(xy) x^{-5/3} F(x). \tag{4.35}$$

From Eqs. (4.32) and (4.35) we have

$$\mathcal{I}_\alpha = c_I u^2 l^{1+\alpha}, \quad c_I = \int_0^\infty dy\, y^\alpha \tilde{B}_L(y). \tag{4.36}$$

The exclusion of $u^2, l^{1+\alpha}$ terms from Eqs. (4.31) and (4.36) yields

$$\mathcal{I}_\alpha = \frac{c_I}{c_F C_K} \Lambda_\alpha. \tag{4.37}$$

From the last connection it follows that time invariance of $\Lambda_\alpha k^\alpha$ implies the invariance of integral $\mathcal{I}_\alpha$. Due to analyticity arguments, the most relevant for study are the choices $\alpha = 2,4,\ldots$. The value of Λ_2 is associated with the so-called Saffman invariant $\mathcal{I}_2$ [20], whereas Λ_4 is proportional to the Loitsyanski integral $\mathcal{I}_4$ [2]. The special case of $d=3$, and $\alpha=2$ is studied in detail in Sec. VI.

It might also be interesting to ask about the initial condition of decay. Since the self-similarity makes the time and wave-number asymptotics interconnected, the initial condition $\lim_{t\to 0} E(k,t) = \Lambda_\alpha k^\alpha$ stems simply from the asymptotic constraint (4.19). From this it follows that the inertial range is not formed for $t \to 0$ [since $l(t) \to 0$]. When t increases, the lower bound of the inertial range moves towards the larger spatial scales and the $E(k,t)$ spectrum attains the shape typical for the three-dimensional developed turbulence.

The remarkable property of the $\Lambda_{d-1} k^{d-1}$ energy spectrum is that it describes $\delta^{(d)}$-correlated (in the real space representation) velocity fluctuations. In addition, any increase of α ($\alpha > d-1$) in the spectrum $\Lambda_\alpha k^\alpha$ induces more pronounced pushing of the energy towards the smaller scales, which means that the spectrum mentioned should be able to model the response of the system on the variety of the stirring regimes. In agreement with the proposed model, the presence of strong stirring near the grid is the characteristic feature of the initial stages of the past grid evolution.

V. NUMERICAL METHOD OF THE CALCULATION OF THE SCALING FUNCTION

At this stage the formulation of the problem of finding F is mathematically completed. Nevertheless, the structure of Eqs. (4.7), (4.17), (4.29), and (4.30) is beyond the scope of standard approaches, especially due to the nonlocal character of the terms $F(q\chi), F(p\chi)$. In addition, the problem is associated with the asymptotic conditions (3.5) and (4.22), which make attempts to find an exact solution untractable. Direct application of the standard numerical approaches seems to be impossible as well.

Our procedure consists in the numerical least-squares minimization of the functional

$$\mathcal{Y}(\overline{b}) \equiv \sum_{\chi \in \mathcal{M}} \chi^{-2}[\hat{L}F_{\mathrm{par}}(\overline{b};\chi) - \psi^{\{F_{\mathrm{par}}(\overline{b};\chi)\}}]^2, \qquad (5.1)$$

which sums up the weighted squared differences $\hat{L}F(\overline{b};\chi) - \psi^{\{F(\overline{b};\chi)\}}$ calculated for the set of the mesh points $\mathcal{M} = \{\chi_1, \chi_2, \ldots\}$ and vector $\overline{b}$ constructed from the variational parameters. The parametrization $F_{\mathrm{par}}(\overline{b};\chi)$ approximating the unknown function $F(\chi)$ [see Eqs. (6.3) and (6.4) below] is suggested to satisfy the asymptotical requirements (3.5) and (4.20).

The form of the $\mathcal{Y}(\overline{b})$ stems simply from a more general functional $\int dk(\partial_t E - T)^2$, attaining the minimum for $\partial_t E = T$ and Eq. (4.2). To reach the minimum numerically, the steepest-descent minimization procedure has been utilized. The minimization of $\mathcal{Y}$ must take into account also Eqs. (4.17), (4.29), and (4.30). For this reason, the basic numerical procedure has been supplemented by iterative steps taking into account the additional equations [in Sec. VI we discuss the effect of further conditions improving the F_{par}]. The convergence of the method towards the minimum is not violated owing to the low sensitivity of $I^{\{F\}}$, C_K, and c_1 to the changes in $\overline{b}$.

To improve the coincidence of $F_{\mathrm{par}}(\overline{b};\chi)$ and $F(\chi)$ at $\chi \gg 1$ Eq. (4.11) has been considered. The asymptote of $F(\chi)$ is assumed to be in the form

$$F(\chi) \simeq F_{\mathrm{par}}(\overline{b};\chi) = 1 - b_h \chi^{\alpha_h} \quad \text{as} \quad \chi \gg 1, \quad \text{and} \quad \alpha_h < 0, \qquad (5.2)$$

which is compatible with the normalization (3.5). This form includes the unknown parameters b_h and α_h. From Eq. (3.9) we obtain

$$\psi^{\{F\}}(\chi)|_{F(\chi)\to 1 - b_h \chi^{\alpha_h}} = -b_h \psi_h(\alpha_h)\chi^{\alpha_h} + \mathcal{O}(\chi^{2\alpha_h}), \qquad (5.3)$$

where

$$\psi_h(\alpha_h) = \tilde{c}_T \int_\Delta dq\ dp\ R_h(q,p,\alpha_h), \qquad (5.4)$$

$$\begin{aligned} R_h(q,p,\alpha_h) = K(p,q)\{&(q^{\alpha_h} + p^{\alpha_h})(pq)^{-2/3-d} \\ &\times [Q(p,q) + Q(q,p)] \\ &-(1+p^{\alpha_h})p^{-2/3-d}Q(p,q) \\ &-(1+q^{\alpha_h})q^{-2/3-d}Q(q,p)\}. \end{aligned}$$

Due to Eq. (4.9), the zeroth-order term $\psi^{\{F\}}(\chi)|_{F=1}$ is not present in the series (5.3). Substitution of Eq. (5.2) into the left-hand side of Eq. (4.7) gives

$$\hat{L}(1 - b_h \chi^{\alpha_h}) = c_1 \chi^{-2/3} - c_1 b_h (1 + c_2 \alpha_h)\chi^{\alpha_h - 2/3}. \qquad (5.5)$$

For $\chi \gg 1$ Eq. (4.7) acquires the form $-b_h \psi_h(\alpha_h)\chi^{\alpha_h} = c_1 \chi^{-2/3}$, leading to the connections

$$b_h = -\frac{c_1}{\psi_h(-2/3)} = \frac{c_h}{\sqrt{C_K}}, \quad c_h = -\frac{\tilde{c}_1}{\psi_h(-2/3)}, \quad \alpha_h = -\frac{2}{3}. \qquad (5.6)$$

VI. PARAMETRIZATION OF THE SCALING FUNCTION FOR $\alpha = 2$ AND $D = 3$

In this section we examine the evolution of decay and scaling behavior in the special case of $\alpha = 2$ and $d = 3$. For these parameters Eqs. (4.23) give the exponents

$$\alpha_l = \frac{2}{5}, \quad \alpha_D = -\frac{11}{5}, \quad \alpha_{\mathcal{E}} = -\frac{6}{5}, \quad c_2 = -\frac{3}{11}, \qquad (6.1)$$

in accord with the earlier result [21] and coinciding with the experimental findings [3]. Due to various remarkable properties of the k^2 spectrum [2,3,13,22] the choice of the $\Lambda_2 k^2$ asymptote of $E(k,t)$ is not motiveless.

Using Eqs. (3.7) and (3.9), we calculate $c_E|_{d=3} = 0.162\,329$ and $c_T|_{d=3} = 6.6\overline{6}$. Hence, from Eqs. (4.29), (4.30), (5.3), (5.6), and (6.1) we obtain

$$\tilde{c}_1 = -0.007\,994, \quad \psi_h(-2/3)|_{d=3} = 0.009\,904,$$
$$c_h = 0.8071, \quad C_K = \frac{1.5}{I^{\{F\}}}. \qquad (6.2)$$

As the simplest one-parameter approximation of $F(\chi)$ we have chosen the von Kármán spectrum having the form

$$F_{\mathrm{par}}(b_0;\chi) = \chi^{11/3}(\chi^2 + b_0^2)^{-11/6}, \quad I^{\{F\}} = 1.261\,96 b_0^{-2/3}. \qquad (6.3)$$

The minimization of the functional $\mathcal{Y}(b_0)$ with respect to b_0 provides the value $b_0 = 1.35$ and corresponding $C_K \simeq 1.45$. Since the parametrization involving the single variational parameter b_0 ignores Eq. (5.6), it plays only the role of a preliminary estimate of $F(\chi)$ (see Fig. 1).

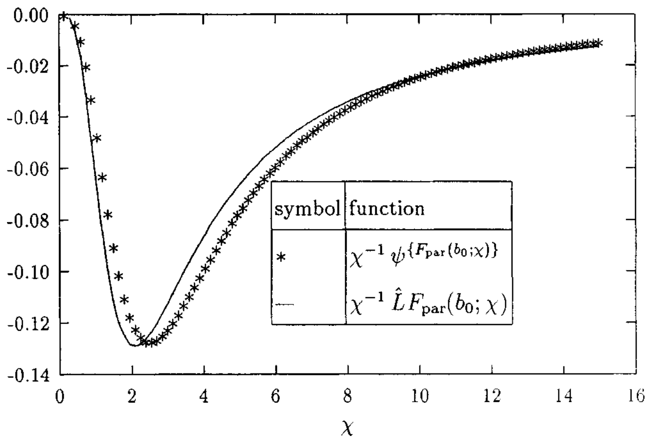


FIG. 1. Functions $\chi^{-1}\hat{L}F_{\rm par}(b_0;\chi)$ and $T(\chi/l,t) \propto \chi^{-1}\psi^{\{F_{\rm par}(b_0;\chi)\}}$ for the parametrization $F^{\rm par}(b_0;\chi)$ [Eq. (6.3)] with $b_0=1.35$. We see that $\{F_{\rm par}(b_0;\chi)\}$ represents only the crude approximation of the solution of the equation $\hat{L}F=\psi^{\{F\}}$.

To account for the discrepancy between the functions $\chi^{-1}\hat{L}F_{\rm par}(b_0;\chi)$ and $\chi^{-1}\psi^{\{F_{\rm par}(b_0;\chi)\}}$, which is obvious near the function minima, and to improve the quality of approximation at very large χ, the refined form of the parametrization has been proposed. We have found that problems for $\chi\sim 2$ can be overcome when the von Kármán form (6.3) is replaced by $\chi^{11/3}(\chi^4+2b_2^2\chi^2+b_1^4)^{-11/12}$. In addition, the requirement (5.2) with $\alpha_h=-2/3$ from Eq. (5.6) leads us to the suggestion of the asymptotically correct prefactor $[1+b_h(b_3^2+\chi^2)^{-1/3}]^{-1}$ with $b_3\neq 0$ preserving the analyticity for $\chi\to 0$. Consequently, the improved parametrization

$$F_{\rm par}(b_1,b_2,b_3,b_h;\chi)=\chi^{11/3}(\chi^4+2b_2^2\chi^2+b_1^4)^{-11/12}\times[1+b_h(b_3^2+\chi^2)^{-1/3}]^{-1} \quad (6.4)$$

contains the variational parameters $\overline{b}=(b_1,b_2,b_3,b_h)$. The minimization of $\mathcal{Y}(b_1,b_2,b_3,b_h)$ carried out for $\mathcal{M}$ involving 50 mesh points uniformly distributed within the interval $0<\chi\leqslant 14$ leads to the numerical estimates

$$b_1=1.578,\quad b_2=0.677,\quad b_3=2.906,\quad b_h=0.642,$$
$$C_K=1.578. \quad (6.5)$$

The coincidence of both sides of Eq. (4.7) is illustrated in Fig. 2. Both Figs. 1 and 2 show two different approximations of $T(k,t)$ in the energy-containing range. Using Eq. (5.2) we find that the lowest infrared correction to the leading Kolmogorov asymptotics $C_K\varepsilon^{2/3}k^{-5/3}$ is of the form $-b_hC_K(\varepsilon/l)^{2/3}k^{-7/3}$.

To compare the theory and experiment we have calculated the longitudinal energy spectrum $E_\parallel$ [2]. The transformation $E\to E_\parallel$ realized with the help of Eq. (4.24) gives

$$E_\parallel\left(\frac{\chi}{l(t)},t\right)=\int_1^\infty dz(z^2-1)z^{-3}E(kz,t)$$
$$=[u(t)]^2 l(t)C_K\chi^{-5/3}\int_0^1 dz(1-z^2)z^{2/3}F(\chi/z). \quad (6.6)$$

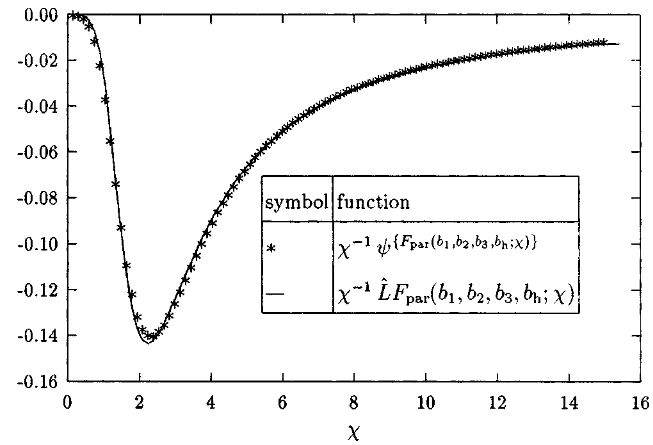


FIG. 2. Coincidence of the functions $\chi^{-1}\hat{L}F_{\rm par}(b_1,b_2,b_3,b_h;\chi)$ and $\chi^{-1}\psi^{\{F_{\rm par}(b_1,b_2,b_3,b_h;\chi)\}}$ calculated for the parameters (6.5) and the set $\mathcal{M}$ involving 50 mesh points.

Figure 3 shows a comparison of the experimental data with our theoretical prediction utilizing the approximation $F(\chi)\simeq F_{\rm par}(b_1,b_2,b_3,b_h;\chi)$. We see that theoretical dependence $E_\parallel(kl)/u^2l$ exhibits particular agreement with the data from the wave-number range between $0.1/l$ and $10/l$, where $E_\parallel/u^2l$ changes approximately two orders of magnitude. Increasing deviations of the data from Kolmogorov's asymptote at high wave numbers should be associated with the viscosity effect.

VII. LIMITATIONS OF THE MODEL

The determination of the consequences of the quasistationary approach is substantial for model applications. We assume that the model introduced should be able to describe a free decay of the turbulence when (i) the von Kármán scale $l(t)$ is the principal and unique length scale determining the self-similar part of the spectra [the occurrence of the scaling

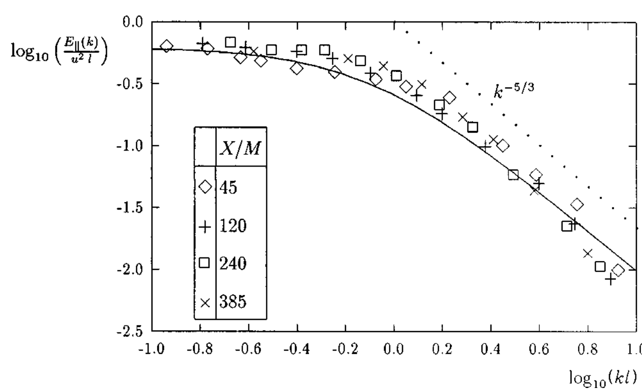


FIG. 3. Comparison of the theoretically predicted form of the longitudinal turbulent energy spectrum $E_\parallel(k)$ and the experimental data in the range $0.1/l\leqslant k\leqslant 10/l$. The experimental points correspond to data taken from [9,23]. They are plotted versus the combination kl in analogy with [9]. The $E_\parallel(k)$ spectrum (6.6) calculated for the parametrization (6.4) is depicted by the continuous curve; the dotted line corresponds to the pure Kolmogorov asymptotic form $k^{-5/3}$. The data were chosen at different past grid distances X related to decay time t; the dimensionless ratio X/M represents the relative distance (time) measured in mesh length units M.

should be expected when $l(t)$ is much larger than the typical viscous scale $\eta(t)$ and much smaller than the outer scale L_0], (ii) the spectral evolution is sufficiently slow and preserving the proportions in the stationary vortex size distribution, and (iii) the molecular viscosity effects are negligible compared to the effects of the vortex inertia. The scales of interest lie in the energy-containing and inertial ranges.

In the next part of the section the impact of above assumptions will be reconsidered quantitatively.

(i) The necessary condition to preserve the self-similarity represented by Eq. (4.2) is that the Kolmogorov scale

$$\eta(t)=\left(\frac{\nu_0^3}{\varepsilon(t)}\right)^{1/4}=\eta_0 t^{(3\alpha+5)/4(\alpha+3)}, \quad \eta_0=\left[\frac{2\nu_0}{(-\alpha_\varepsilon)l_0^{2/3}d}\right]^{3/4} \tag{7.1}$$

and the von Kármán scale $l(t)$ are widely separated:

$$l(t)\gg\eta(t). \tag{7.2}$$

The additional condition [2]

$$l(t)\ll L_0 \tag{7.3}$$

determines the bounds of the spectral representation. If the Reynolds number on the scale l is defined as

$$\mathrm{Re}_l(t)=\frac{u(t)l(t)}{\nu_0}=\left(\frac{l(t)}{\eta(t)}\right)^{4/3}=\left(\frac{l_0}{\eta_0}\right)^{4/3}t^{(1-\alpha)/(3+\alpha)}, \tag{7.4}$$

then the condition (7.2) is equivalent to the requirement $\mathrm{Re}_l(t)\gg 1$. As it follows from Eq. (7.4), for $\alpha\geq 1$, $\mathrm{Re}_l(t)$ decreases with t. Therefore, the model studied may fail to give an accurate description of the very late time stages of decay. From Eqs. (4.12) and (4.23) it follows that both criteria (7.2) and (7.3) can be fulfilled simultaneously if

$$t\ll t_{\max}=\min\left\{\left(\frac{L_0}{l_0}\right)^{(\alpha+3)/2},\left(\frac{l_0}{\eta_0}\right)^{4(\alpha+3)/3(\alpha-1)}\right\}. \tag{7.5}$$

Let $k_E(t)$ denotes the coordinate of the maximum of $E(k,t)$ with respect to variable k [i.e. $E(k,t)\leq E(k_E(t),t)$]; $k_E(t)$ can be associated with the lower bound of the inertial range defined by the inequalities $k_E(t)\ll k\ll 1/\eta(t)$. Formally, $k_E(t)=\kappa_E/l(t)=(\kappa_E/l_0)t^{-\alpha_l}$, with the expectation $\kappa_E=\mathcal{O}(1)$. For the parametrization (6.4) we have found that $\kappa_E\simeq 1.68$.

(ii) The principal dynamical characteristic of the stationary system is the time scale $1/\nu_0 k^2$. It is involved in the exponentials of the bare propagators [see Eq. (A5)]. The application of the quasistationary approach is justified if the free decay process does not require one to change the form of the stationary propagators. The precondition of the last requirement is that the viscous damping is much faster than the overall decay, i.e.,

$$t\gg t_{\min}\equiv\max_{k\eta\leq 1\leq kL_0}\left\{\frac{1}{\nu_0 k^2}\right\}\simeq\frac{L_0^2}{\nu_0}. \tag{7.6}$$

(iii) To guarantee the dominance of the inertia effects, we assume that

$$\nu(k,t)\gg\nu_0, \tag{7.7}$$

where $\nu(k,t)=\nu(k)|_{\overline{D}\to\overline{D}(t)}$ is the quasistationary form of the eddy turbulent viscosity. The pure stationary form

$$\nu(k)\equiv\left(\frac{\overline{D}}{g_*}\right)^{1/3}k^{-4/3} \tag{7.8}$$

was predicted by RG theory [12,14]. The condition (7.7) is equivalent to

$$\chi\ll[\mathrm{Re}_l(t)]^{3/4}\left(\frac{C_K}{c_E}\right)^{3/8}g_*^{-1/4}. \tag{7.9}$$

For the parameters (3.7) and (6.5) we obtain that the numerical value of the prefactor from the last expression is $(C_K/c_E)^{3/8}g_*^{-1/4}=0.5825$. Using Eq. (4.23) for $\alpha=2$, we have obtained $l(t)\propto t^{0.4}$, $\eta(t)\propto t^{0.55}$, and $\mathrm{Re}_l(t)\propto t^{-0.2}$, whereas the substitution of $\alpha=4$ yields $l(t)\propto t^{0.286}$, $\eta(t)\propto t^{0.607}$, and $\mathrm{Re}_l(t)\propto t^{-0.428}$. Comparing these two decay regimes one can conclude that the time interval where the scaling (4.2) should be expected is wider in the case of decay evolving via the Λ_2 invariant.

VIII. CONCLUSION

We have presented a model that enables us to calculate the scaling forms of the energy spectrum and energy transfer of decaying turbulence. The aim of the work has been to study the universal aspects of the isotropic and homogeneous turbulence of the energy-containing and adjacent inertial range, where decay laws approach the power form.

The initial point of the study includes the results obtained for the stationary model of randomly forced turbulence with the extended form of random forcing. To describe the free decay of the past grid turbulence, the results of the stationary model were modified in the framework of the quasistationary approach. An *a posteriori* analysis has imposed the limitations $t_{\min}\leq t\leq t_{\max}$ representing the preconditions for the applicability of the quasistationary approach to decay statistics. Since the small-k behavior of the spectrum in the Navier-Stokes turbulence has not been firmly established up to now, we have adopted for this aim Saffman's hypothesis. Fixing the parameter α is sufficient for the determination of the scaling function F, which is free of other adjustable parameters.

The computation of F has been performed for $d=3$ and $\alpha=2$. It has been recognized that the shape of the functions $E(\chi/l)$ and $T(\chi/l,t)$ exhibits qualitative agreement with the canonical expectations for the energy-containing range [3]. The approximately calculated longitudinal energy spectrum shows promising agreement with the available experimental data. Due to insufficient accuracy and reproducibility of the current data, the quantitative testing of the third-order statistics is a difficult task.

In this paper we have done more than just derive and present a model of decay; there is a proposal for further work towards the understanding of the universal aspects of turbulence. Most importantly, the presence of the multiplicative (intermittency) correction to the $k^{-5/3}$ asymptote of $E(k,t)$ could be recognized as well and the construction of the equation for the scaling function could be studied in more detail. In addition, one could analyze a more complex form of the parametrization for different d ($d>2$) and α to obtain more comprehensive numerical results.

ACKNOWLEDGMENTS

M.H. and D.H. thank Dr. J. Honkonen for a critical reading of the manuscript. This work was supported by Fundamental Research Russian Fund, International Scientific Fund (Grant No. 96-02-17-033) and by Slovak Grant Agency for Science (Grant No. 2/4171/97).

APPENDIX A: STATIONARY SCALING FORM OF THE ENERGY TRANSFER

To help the reader understand the origin of the formulas (3.9)–(3.12), the derivation of the triple velocity correlation function [15] is presented in this appendix in a simplified and shortened form. For the velocity field evolution described by Eqs. (3.1) and (3.3) the Martin-Siggia-Rose formalism [24] permits us to find the functional integral expressions for the expectation values of generating functional with the effective action

$$\mathcal{S}=\frac{1}{2}\sum_{j,s}^{d}\int d^d\mathbf{x}_1 d^d\mathbf{x}_2 d\tau[\overline{v}_j(\mathbf{x}_1,\tau)\mathcal{P}_{js}(\boldsymbol{\nabla}_{\mathbf{x}_{12}})\times\mathcal{C}(|\mathbf{x}_{12}|)\overline{v}_s(\mathbf{x}_2,\tau)]+\int d^d\mathbf{x}\, d\tau\{\overline{\mathbf{v}}(\mathbf{x},\tau)\cdot[-\partial_\tau\mathbf{v}-(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}+\nu_0\nabla^2\mathbf{v}]_{(\mathbf{x},\tau)}\}. \tag{A1}$$

Here $\overline{\mathbf{v}}$ is the transverse auxiliary field (independent of $\mathbf{v}$), which has been introduced in the description by the transformation of the initial stochastic problem (3.1)–(3.3) in functional form. Following Ref. [14], single-time triple correlation functions can be expressed through the functional derivatives as

$$\langle v_\mu(\mathbf{x}_1)v_\beta(\mathbf{x}_2)v_\gamma(\mathbf{x}_3)\rangle=\frac{1}{\mathcal{Z}}\left|\frac{\delta^3\mathcal{Z}}{\delta\sigma_\mu(\mathbf{x}_1,\tau)\,\delta\sigma_\beta(\mathbf{x}_2,\tau)\,\delta\sigma_\gamma(\mathbf{x}_3,\tau)}\right|_{\sigma=0,\overline{\sigma}=0}, \tag{A2}$$

where the generating functional [12]

$$\mathcal{Z}\{\boldsymbol{\sigma},\overline{\boldsymbol{\sigma}}\}=\int\mathcal{D}\mathbf{v}\int\mathcal{D}\overline{\mathbf{v}}\exp\left[\mathcal{S}+\int d^d\mathbf{x}\,d\tau(\boldsymbol{\sigma}\cdot\mathbf{v}+\overline{\boldsymbol{\sigma}}\cdot\overline{\mathbf{v}})_{(\mathbf{x},\tau)}\right] \tag{A3}$$

can be calculated perturbatively using the functional formula [25]

$$\mathcal{Z}\{\boldsymbol{\sigma},\overline{\boldsymbol{\sigma}}\}=\exp\left[\frac{1}{2}\sum_{m,s,n}^{d}\int d^d\mathbf{x}\,d\tau\left\{\frac{\delta}{\delta\sigma_s(\mathbf{x},\tau)}\frac{\delta}{\delta\sigma_n(\mathbf{x},\tau)}\times V_{msn}(-i\boldsymbol{\nabla})\frac{\delta}{\delta\overline{\sigma}_m(\mathbf{x},\tau)}\right\}\right]\times\exp\left[\frac{1}{2}\sum_{j,l}^{d}\int d^d\mathbf{x}_1 d\tau_1 d^d\mathbf{x}_2 d\tau_2\times\{\sigma_j(\mathbf{x}_1,\tau_1)G_{jl}(\mathbf{x}_{12},\tau_{12})\sigma_l(\mathbf{x}_2,\tau_2)+\sigma_j(\mathbf{x}_1,\tau_1)\overline{G}_{jl}(\mathbf{x}_{21},\tau_{21})\overline{\sigma}_l(\mathbf{x}_2,\tau_2)+\overline{\sigma}_j(\mathbf{x}_1,\tau_1)\overline{G}_{jl}(\mathbf{x}_{12},\tau_{12})\sigma_l(\mathbf{x}_2,\tau_2)\}\right] \tag{A4}$$

involving the bare propagators

$$\overline{G}_{\mu s}(\mathbf{x},\tau)=\int\frac{d^d\mathbf{k}}{(2\pi)^d}\,e^{i\mathbf{k}\cdot\mathbf{x}}\overline{G}^0_{\mu s}(\mathbf{k},\tau),$$

$$G_{\mu s}(\mathbf{x},\tau)=\int\frac{d^d\mathbf{k}}{(2\pi)^d}\,e^{i\mathbf{k}\cdot\mathbf{x}}G^0_{\mu s}(\mathbf{k},\tau),$$

$$\overline{G}^0_{\mu s}(\mathbf{k},\tau)=\mathcal{P}_{\mu s}(\mathbf{k})\,\theta(-\tau)e^{\nu_0 k^2\tau},$$

$$G^0_{\mu s}(\mathbf{k},\tau)=\frac{\overline{D}\mathcal{F}(kl)}{2\nu_0 k^{2+d}}\,\mathcal{P}_{\mu s}(\mathbf{k})e^{-\nu_0 k^2|\tau|}, \tag{A5}$$

and the three-point vertices $V_{msn}(\mathbf{p})=i(\delta_{ms}p_n+\delta_{mn}p_s)$ and $V_{msn}(-i\boldsymbol{\nabla})=(\delta_{ms}\partial_{x_n}+\delta_{mn}\partial_{x_s})$; $\theta(-\tau)$ is the usual step function. The sequence of integrals generated in a perturbative way from Eq. (A4) may be represented pictorially by the Feynman diagrams. Superficial diagrammatic divergences arising in them are the classical artifacts of the perturbative treatment and the quantum field RG approach [14] is conceptually related to their elimination.

It is well known that recursive RG relations derived for the action (A1) in the frame of the ϵ-expansion scheme exhibit the infrared stable fixed point associated with the Kolmogorov scaling. The knowledge of the fixed point parameters allows us to find the leading infrared asymptotics of the energy spectrum and energy transfer. The complete calculation of $\mathcal{F}$ is beyond the scope of the RG theory, which is restricted to the analysis in the massless limit $\mathcal{F}(\chi)|_{\chi\to\infty}$. This conventional analysis is based on the assumption that the next to leading order asymptote of the function $\mathcal{F}(kl)$ [see Eqs. (3.5) and (5.2)] is free of terms affecting the $k^{-5/3}$ tail of $E(k)$.

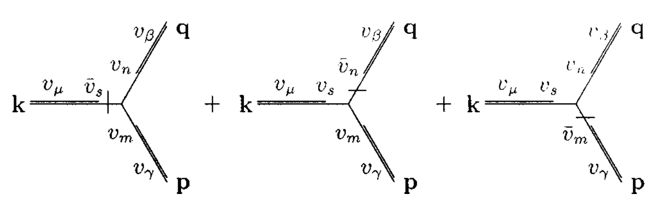
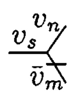


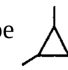
FIG. 4. Sum of the lowest-order diagrams contributing in the correlation function $\mathcal{T}_{\mu\beta\gamma}(\mathbf{k},\mathbf{q},\mathbf{p})$.

The explicit form of the equal-time triple correlation function $\langle v_\mu v_\beta v_\gamma\rangle$ may be more easily analyzed in wave-vector space. Due to translational invariance we assume

$$\langle v_\mu(\mathbf{x}_1)v_\beta(\mathbf{x}_2)v_\gamma(\mathbf{x}_3)\rangle = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\int \frac{d^d\mathbf{p}}{(2\pi)^d}\, e^{i(\mathbf{k}\cdot\mathbf{x}_1+\mathbf{q}\cdot\mathbf{x}_2+\mathbf{p}\cdot\mathbf{x}_3)} \times(2\pi)^d\delta^{(d)}(\mathbf{k}+\mathbf{q}+\mathbf{p})\mathcal{T}_{\mu\beta\gamma}(\mathbf{k},\mathbf{q},\mathbf{p}). \tag{A6}$$

The Fourier transform $\mathcal{T}_{\mu\beta\gamma}$ of the correlation function can be approximated by means of the sum of diagrams in Fig. 4. The isotropy assumption implies that the result of the summation has to be symmetric with respect to the permutation of $\mathbf{k}$, $\mathbf{q}$, and $\mathbf{p}$ vectors. The triple vertex  is equivalent to $V_{msn}(\mathbf{p})$ in the notation used. The diagrammatic double lines $\mathbf{k}$; $\tau_0 \underset{}{\overset{v_\mu\,\bar{v}_s}{=\!=\!=\!+}} \tau_1 \equiv \overline{G}^R_{\mu s}(\mathbf{k},\tau_0-\tau_1)$ and $\mathbf{k}$; $\tau_0 \overset{v_\mu\,v_s}{=\!=\!=} \tau_1 \equiv G^R_{\mu s}(\mathbf{k},\tau_0-\tau_1)$ are associated with the renormalized propagators

$$\overline{G}^R_{\mu s}=\overline{G}^0_{\mu s}|_{\nu_0\to\nu(k)}, \quad G^R_{\mu s}=G^0_{\mu s}|_{\nu_0\to\nu(k)}, \tag{A7}$$

which can be obtained from the bare propagators by replacing ν_0 by $\nu(k)$, where $\nu(k)$ is the eddy viscosity [see Eq. (7.8)]. The impact of the renormalized propagators is equivalent to the resummation of the relevant ''self-energy'' contributions. It should be emphasized that no vertex corrections of the type  are needed in the expression for the correlation function $\mathcal{T}$ in our approximation. The justification of this simplification was given in [15], where the irrelevance of the overall effect of all the vertex loops was demonstrated within the one-loop order of the *short-distance expansion*. When the diagrammatic sum (see Fig. 4) written in the algebraic form is modified with the help of Eq. (7.8), it gives

$$\begin{aligned}\mathcal{T}_{\mu\beta\gamma}(\mathbf{k},\mathbf{q},\mathbf{p}) &= \sum_{s,n,m}^{d}\int_{-\infty}^{\infty} d\tau[\overline{G}^R_{\mu s}(\mathbf{k},\tau)G^R_{\beta n}(\mathbf{q},\tau)G^R_{\gamma m}(\mathbf{p},\tau)V_{snm}(\mathbf{k})\\ &\quad + G^R_{\mu s}(\mathbf{k},\tau)\overline{G}^R_{\beta n}(\mathbf{q},\tau)G^R_{\gamma m}(\mathbf{p},\tau)V_{nms}(\mathbf{q}) + G^R_{\mu s}(\mathbf{k},\tau)G^R_{\beta n}(\mathbf{q},\tau)\overline{G}^R_{\gamma m}(\mathbf{p},\tau)V_{msn}(\mathbf{p})]\\ &= \frac{\overline{D}^2}{4}\sum_{s,n,m}^{d}\frac{\mathcal{P}_{\mu s}(\mathbf{k})\mathcal{P}_{\beta n}(\mathbf{q})\mathcal{P}_{\gamma m}(\mathbf{p})}{(\nu(k)k^2+\nu(q)q^2+\nu(p)p^2)}\Bigg(V_{snm}(\mathbf{k})\frac{\mathcal{F}(ql)}{\nu(q)}\frac{\mathcal{F}(pl)}{\nu(p)}(qp)^{-d-2}\\ &\quad + V_{nms}(\mathbf{q})\frac{\mathcal{F}(pl)}{\nu(p)}\frac{\mathcal{F}(kl)}{\nu(k)}(pk)^{-d-2} + V_{msn}(\mathbf{p})\frac{\mathcal{F}(kl)}{\nu(k)}\frac{\mathcal{F}(ql)}{\nu(q)}(kq)^{-d-2}\Bigg)\\ &= \frac{g_*\overline{D}}{4(k^{2/3}+q^{2/3}+p^{2/3})}\sum_{s,n,m}^{d}\mathcal{P}_{\mu s}(\mathbf{k})\mathcal{P}_{\beta n}(\mathbf{q})\mathcal{P}_{\gamma m}(\mathbf{q})[V_{snm}(\mathbf{k})\mathcal{F}(ql)\mathcal{F}(pl)(qp)^{-d-2/3}\\ &\quad + V_{nms}(\mathbf{q})\mathcal{F}(pl)\mathcal{F}(kl)(pk)^{-d-2/3} + V_{msn}(\mathbf{p})\mathcal{F}(kl)\mathcal{F}(ql)(kq)^{-d-2/3}].\end{aligned} \tag{A8}$$

It follows from Eqs. (2.2) and (A6) that energy transfer can be simply associated with the triple correlation function via the relation

$$T(k)=\frac{k^{d-1}S_d}{2(2\pi)^{2d}}\sum_{\mu,\beta,\gamma}^{d}\int d^d\mathbf{q}\; V_{\mu\beta\gamma}(-\mathbf{k}) \times\mathcal{T}_{\mu\beta\gamma}(\mathbf{k},\mathbf{q},\mathbf{p})|_{\mathbf{p}=-\mathbf{k}-\mathbf{q}}. \tag{A9}$$

Further analysis of the structure of the integrand from Eq. (A9) [defined by Eq. (A8)] requires the calculation of the expressions of the type

$$\sum_{\mu,s,\beta,n,\gamma,m}^{d}\mathcal{P}_{\mu s}(\mathbf{k})\mathcal{P}_{\beta n}(\mathbf{q})\mathcal{P}_{\gamma m}(\mathbf{p})\begin{pmatrix}V_{nms}(\mathbf{q})\\ V_{msn}(\mathbf{p})\\ V_{snm}(\mathbf{k})\end{pmatrix}V_{\mu\beta\gamma}(-\mathbf{k}) = \frac{k^2[k^2q^2-(\mathbf{k}\cdot\mathbf{q})^2]}{q^2p^2}\begin{pmatrix}-Q(p/k,q/k)\\ -Q(q/k,p/k)\\ Q(p/k,q/k)+Q(q/k,p/k)\end{pmatrix} \tag{A10}$$

for $\mathbf{p}=-\mathbf{k}-\mathbf{q}$. [The function Q is defined by Eq. (3.13)]. From the structure of Eq. (A10) it follows that Eq. (A8) can be written as a function of the arguments k, q, and $\mathbf{k}\cdot\mathbf{q}$:

$$T(k)=\frac{k^{d-1}S_d}{2(2\pi)^{2d}}\int d^d\mathbf{q}\ \Phi(k,q,\mathbf{k}\cdot\mathbf{q}), \tag{A11}$$

where

$$\Phi(k,q,\mathbf{k}\cdot\mathbf{q})=\sum_{\mu,\beta,\gamma}^{d} V_{\mu\beta\gamma}(-\mathbf{k})\mathcal{T}_{\mu\beta\gamma}(\mathbf{k},\mathbf{q},\mathbf{p})|_{\mathbf{p}=-\mathbf{k}-\mathbf{q}}. \tag{A12}$$

In the spherical coordinates, the integral (A11) can be written as

$$\int d^d\mathbf{q}\ \Phi(k,q,\mathbf{k}\cdot\mathbf{q})=S_{d-1}\int_0^\infty dq\ q^{d-1}\int_0^\pi d\phi \times(\sin\phi)^{d-2}\Phi(k,q,kq\ \cos\phi), \tag{A13}$$

where ϕ labels the angle between the vectors $\mathbf{k}$ and $\mathbf{q}$. Now, to obtain the energy transfer in the form symmetric in variables p and q, we are introducing the transformation $(q,\phi)\to(q',p')$, where $q'=q/k$ and

$$p'=\sqrt{1+2(q/k)\cos\phi+(q/k)^2}.$$

It transforms the spherical coordinates (q,ϕ) to dimensionless bipolar coordinates (q',p'). Using

$$dq\ d\phi=dq'dp'\left|\det\left(\frac{\partial(q,\phi)}{\partial(q',p')}\right)\right|=dq'dp'\ \frac{p'k}{q'\sin\phi},$$

$$\cos\phi=\frac{(p')^2-(q')^2-1}{2q'},$$

$$\sin\phi=\frac{\sqrt{2[(q')^2+(q'p')^2+(p')^2]-[1+(q')^4+(p')^4]}}{2q'}, \tag{A14}$$

we obtain

$$\begin{aligned}\int d^d\mathbf{q}\ \Phi(k,q,\mathbf{k}\cdot\mathbf{q}) &=S_{d-1}2^{3-d}k^d\\ &\times\int_\Delta dq'dp'p'q'[2[(q')^2+(q'p')^2+(p')^2]\\ &-[1+(q')^4+(p')^4]]^{(d-3)/2}\\ &\times\Phi\left(k,kq',\frac{1}{2}[(p'k)^2-(q'k)^2-k^2]\right).\end{aligned} \tag{A15}$$

The system of equations (3.10)–(3.13) can be obtained from Eq. (A15) by using the formal replacements $p'\to p$ and $q'\to q$.

APPENDIX B: NUMERICAL CALCULATION OF THE INTEGRALS

In this appendix algebraic manipulations are presented, which were successfully applied to calculate numerically $\psi^{\{F_{\rm par}(b_1,b_2,b_3,b_h;\chi)\}}(\chi)$ by using Eq. (3.9). To explain the way to overcome the unfavorable numerical effects, it is useful to decompose the integration domain Δ [defined by Eq. (3.14)]. The decomposition introduces the elementary subdomains (see Fig. 5)

$$\Delta=\Delta_v^1\cup\Delta_v^2\cup\Delta_\varphi,$$

$$\Delta_\varphi=\{(q,p);|1-q|\le p\le q\},$$

$$\Delta_v^1=\{(q,p);0\le q\le 1/2,1-q\le p\le 1+q\}, \tag{B1}$$

$$\Delta_v^2=\{(q,p);1/2\le q\le p\le 1+q\}.$$

From the singularity formed by the kernel (3.12) it might be deduced that any numerical procedure covering the full domain Δ (3.10) has to eliminate the effects induced by poles: $(q=0,\ p=1)$ and $(q=1,\ p=0)$. Taking into account the symmetry of the integrand (3.11)

$$R(q,p;\chi)=R(p,q;\chi), \tag{B2}$$

we see that the overall integration domain Δ can be reduced to $\Delta_v^1\cup\Delta_v^2$. After this step we obtain the formula

$$\int_\Delta dq\ dp\ R(q,p;\chi)=2\sum_{j=1,2}\int_{\Delta_v^j}dq\ dp\ R(q,p;\chi), \tag{B3}$$

in which the problem connected with the single pole $(q=0,\ p=1)$ remains. Further appropriate transformations

$$q=\frac{y}{2},\quad p=\frac{1-y+\zeta(w,y)}{2},\quad \zeta(w,y)=1+2wy\quad \text{for } \Delta_v^1, \tag{B4}$$

$$q=\frac{1}{2y},\quad p=\frac{\zeta(w,y)}{2y}\quad \text{for } \Delta_v^2 \tag{B5}$$

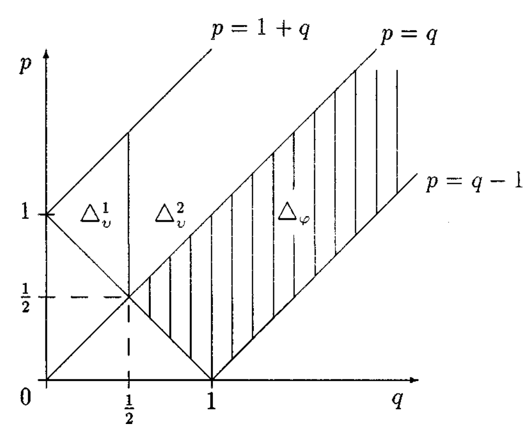


FIG. 5. Schematic decomposition of the integration domain Δ into its subdomains Δ_v^1, Δ_v^2, and Δ_φ.

are used to map Δ_v^1 and Δ_v^2 onto the unit square $\langle 0,1\rangle \times \langle 0,1\rangle$. Taking into account the change of the integration measures we obtain

$$\int_{\Delta_v^1} dq\, dp\, R(q,p;\chi) = \int_0^1 \int_0^1 dy\, dw\, \frac{y}{2} R\left(\frac{y}{2}, \frac{1-y+\zeta(w,y)}{2};\chi\right), \tag{B6}$$

$$\int_{\Delta_v^2} dq\, dp\, R(q,p;\chi) = \int_0^1 \int_0^1 dy\, dw\, \frac{1}{2y^2} R\left(\frac{1}{2y}, \frac{\zeta(w,y)}{2y};\chi\right). \tag{B7}$$

It follows from Eq. (B6) that the effect of the transformation (B4) is constructive since the Jacobian $|\det[\partial(q,p)/\partial(y,w)]| = y/2$ cancels the pole $1/y$ concerned with the kernel $K[p(y,w),y/2]$. The second transformation (B5) seems to be less useful because of the production of many apparent singularities of the integrand (B7). However, the potential numerical complications can be simply eliminated using the transcription

$$\frac{1}{2y^2} R\left(\frac{1}{2y}, \frac{\zeta(w,y)}{2y};\chi\right) = R_1(w,y;\chi) - y^{-2/3} R_2(w,y;\chi). \tag{B8}$$

The functions R_1 and R_2 are defined by

$$R_1(w,y;\chi) = y^d K_0(w,y)\, \frac{F_0(\zeta(w,y)\chi,2y) F_0(\chi,2y)}{[\zeta(w,y)/2]^{d+2/3}} \times [Q_1(w,y)+Q_2(w,y)],$$

$$R_2(w,y;\chi) = K_0(w,y) F(\chi) \left(\frac{F_0(\zeta(w,y)\chi,2y)}{[\zeta(w,y)]^{d+2/3}} \times Q_1(w,y) + F_0(\chi,2y) Q_2(w,y) \right),$$

$$K_0(w,y) = \frac{2^{d+1/3}\{(1-w^2)[\zeta(w,y)+y^2(w^2-1)]\}^{(d-1)/2}}{\zeta(w,y)\{1+(2y)^{2/3}+[\zeta(w,y)]^{2/3}\}} = \frac{y^{d-11/3}}{2^{7/3-d}} K\left(\frac{\zeta(w,y)}{2y}, \frac{1}{2y}\right),$$

$$Q_1(w,y) = 4y^3 Q\left(\frac{\zeta(w,y)}{2y}, \frac{1}{2y}\right) = (d-2+5w^2)y + w + 4wy^2[2w^2-1+wy(w^2-1)],$$

$$Q_2(w,y) = 4y^3 Q\left(\frac{1}{2y}, \frac{\zeta(w,y)}{2y}\right) = (d-2-w^2)y - w + 4(d-1)wy^2(1+wy),$$

$$F_0(x,x_1) \equiv F_{\rm par}\left(b_1,b_2,b_3,b_h;\frac{x}{x_1}\right) = F_{\rm par}(b_1x_1,b_2x_1,b_3x_1,b_hx_1^{2/3};x). \tag{B9}$$

The essential property allowing the elimination of the numerically difficult terms in R_1 and R_2 is the homogeneity of the parametrization $F_{\rm par}(b_1\xi,b_2\xi,b_3\xi,b_h\xi^{2/3};\chi\xi) = F_{\rm par}(b_1,b_2,b_3,b_h;\chi)\ \forall \xi$ [see Eq. (6.4)]. In addition, the replacement $y \to y^3$, which eliminates the integrable singularity $y^{-3/2}$ arising in Eq. (B8), leads to the formula

$$\int_{\Delta_v^2} dq\, dp\, \frac{1}{2y^2} R(q,p;\chi) = \int_0^1 \int_0^1 dy\, dw [R_1(w,y;\chi) - 3R_2(w,y^3;\chi)]. \tag{B10}$$

We conclude this appendix by noting that the convenient way to calculate $I^{\{F\}}$ [see Eq. (4.17)] is to use the formula

$$I^{\{F\}} = \int_0^1 dx\, x^{-1/3}[F_0(1,x) + x^{-4/3} F_{\rm par}(b_1,b_2,b_3,b_h;x)]. \tag{B11}$$

[1] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **31**, 538 (1941).
[2] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics: Mechanics of Turbulence* (MIT Press, Cambridge, MA, 1975), Vol. 2.
[3] W. D. McComb, *The Physics of Fluid Turbulence* (Clarendon, Oxford, 1990).
[4] C. De Dominicis and P. C. Martin, Phys. Rev. A **19**, 419 (1979).
[5] H. W. Wyld, Ann. Phys. (N.Y.) **14**, 143 (1961).
[6] V. Yakhot and S. A. Orszag, J. Sci. Comput. **1**, 3 (1986); Phys. Rev. Lett. **57**, 1722 (1986).
[7] W. P. Dannevik, V. Yakhot, and S. A. Orszag, Phys. Fluids **30**, 2021 (1987).
[8] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasil'ev, Zh. Eksp. Teor. Fiz. **95**, 1272 (1989) [Sov. Phys. JETP **68**, 733 (1989)].
[9] W. M. George, Phys. Fluids A **4**, 1492 (1992).
[10] M. Lesieur and D. Schertzer, J. Mec. **17**, 609 (1978).
[11] G. I. Taylor, Proc. R. Soc. London, Ser. A **164**, 476 (1938).
[12] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasiliev, Usp. Fiz. Nauk **166**, 1257 (1996).
[13] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).
[14] L. Ts. Adzhemyan, A. N. Vasil'ev, and Yu. M. Pis'mak, Teor. Mat. Fiz. **57**, 286 (1983).

[15] L. Ts. Adzhemyan, S. V. Borisenok, and V. I. Girina, Teor. Mat. Fiz. **105**, 450 (1995).
[16] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1989).
[17] J. D. Fournier and U. Frisch, Phys. Rev. A **17**, 747 (1978).
[18] S. A. Orszag, in *Fluid Dynamics*, 1973 Les Houches Lectures, edited by R. Balian and J. L. Peube (Gordon and Breach, New York, 1977), p. 235.
[19] Th. von Kármán and L. Howarth, Proc. R. Soc. London, Ser. A **164**, 192 (1938).
[20] P. G. Saffman, J. Fluid Mech. **27**, 581 (1967).
[21] P. G. Saffman, Phys. Fluids **10**, 1349 (1967).
[22] R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).
[23] G. Comte-Bellot and S. Corrsin, J. Fluid Mech. **48**, 273 (1971).
[24] P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A **8**, 423 (1973).
[25] A. A. Slavnov and L. D. Fadejev, *Vvedenje v Kvantovuju Teoriju Kalibrovochnych Polej* (Nauka, Moscow, 1978).